数理解析研究所講究録 第2113巻 2019年 113-122

Characterization of operator convex functions by certain operator inequalities

Yukihiro Tsurumi
Graduate school of science and engineering,
Ritsumeikan University

1 Introduction

In this article we consider the extension of the following observation. Let σ be an operator mean.

1. If $A\sigma B \geq A\nabla B$ for any $A, B \in B(H)^{++}$, then $\sigma = \nabla$.

2. If $A\sigma B \leq A!B$ for any $A, B \in B(H)^{++}$, then $\sigma =!$.

$\Longrightarrow$

Let $\lambda \in [0,1]$ and ψ be a non-negative continuous function on $[0, \infty)$.

1. If $\psi(A)\sigma\psi(B) \geq \psi(A\nabla_\lambda B)$ for any $A, B \in B(H)^{++}$, then $\sigma = \nabla_\lambda$?

2. If $\psi(A)\sigma\psi(B) \leq \psi(A!_\lambda B)$ for any $A, B \in B(H)^{++}$, then $\sigma =!_\lambda$?

The reason for considering these operator mean inequalities is based on trying to evaluate the relative entropy in quantum information theory. Furuichi introduced the following relative entropy in 2012.

Definition 1.1 ([3]). *For a continuous and strictly monotone function ψ on $(0, \infty)$ and two probability distributions $\{p_1, ..., p_n\}$, $\{q_1, ..., q_n\}$ with $p_j, q_j > 0$ for all $j = 1, ..., n$ the Tsallis quasilinear relative entropy is defined by*

$$D_r^\psi\left(p_1, ..., p_n || q_1, ..., q_n\right) := -\ln_r \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{q_j}{p_j}\right)\right)$$

where $\ln_r(x) = \dfrac{x^{1-r} - 1}{1-r}$ $r \in [0,1)$, $\log(x) = \lim_{r \to 1} \ln_r(x)$.

- When $\psi(x) = \ln_r(x)$,

$$\textit{Tsallis relative entropy } D_r^{\ln_r} = -\sum_{j=1}^{n} p_j \left(\ln_r \left(\frac{q_j}{p_j} \right) \right).$$

- When $\psi(x) = x^{1-s}$ and $r = 1$,

$$\textit{Rényi relative entropy } D_1^{x^{1-s}} = \frac{1}{s-1} \log \left(\sum_{j=1}^{n} p_j^s q_j^{1-s} \right).$$

We quantize this relative entropy by setting positive operators (matrices) A and B like the following

$$A = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix}, B = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{bmatrix}$$

and

$$\psi\left(A^{-1/2}BA^{-1/2}\right) = \begin{bmatrix} \psi\left(\frac{q_1}{p_1}\right) & & \\ & \ddots & \\ & & \psi\left(\frac{q_n}{p_n}\right) \end{bmatrix},$$

then we can get like the following formulation

$$D_r^{\psi}\left(A||B\right) = -\ln_r \psi^{-1} \left(\mathrm{Tr} A^{1/2} \psi \left(A^{-1/2} B A^{-1/2} \right) A^{1/2} \right)$$

It is not easy to evaluate the whole of this formulated $D_r^{\psi}\left(A||B\right)$ directly. So in this paper, we characterize only the part $A^{1/2}\psi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$ in $D_r^{\psi}\left(A||B\right)$ using the operator inequality. Furthermore, We give the characterization of operator convex from the operator mean.

Ando and Hiai gave a following characterization of an operator monotone decreasing function by means of certain operator inequalities in ([1]). The main result of this article is based on the following Ando-Hiai results. To get this result, for a non-negative continuous function ψ on $(0,\infty)$ and $\lambda \in (0,1)$, we consider the set $\Gamma_\lambda(\psi)$ of operator means σ such that the inequality

$$\psi(A \nabla_\lambda B) \leq \psi(A) \sigma \psi(B)$$

holds for all $A, B \in B(H)^{++}$.

Theorem ([1])**.** *Let ψ be a continuous non-decreasing function on $[0,\infty)$ such that $\psi(0)=0$ and $\psi(1)=1$. If a symmetric operator mean σ satisfies*

$$\psi(A\nabla B) \leq \psi(A)\sigma\psi(B)$$

for any $A, B \in B(H)^{++}$, then $\sigma = \nabla$.

In this article, we only gives the rough proof for each propositions in Section 3,4,5. The details of each proof are given in ([12]).

2 Fundamental definitions and notations

A self-adjoint operator A acting on a Hilbert space H is said to be *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in H$, we denote this by $A \geq 0$. Let $B(H)^+$ be the set of all positive operator on H, and let $B(H)^{++}$ be the set of all positive invertible operator on H. Let f be a continuous real-valued function on $(0,\infty)$. f is called *n-monotone* if positive invertible operators $A, B \in M_n(\mathbf{C})$ with $A \leq B$, then $f(A) \leq f(B)$. f is called *operator monotone* if for any $n \in \mathbf{N}$ f is n-monotone. Similarly, f is called *n-convex* if $f(\lambda A+(1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$ for positive invertible operators $A, B \in M_n(\mathbf{C})$ and for any $\lambda \in [0,1]$. f is called *operator convex* if for any $n \in \mathbf{N}$ f is n-convex. As for typical examples of them, power function t^s on $(0,\infty)$ is operator monotone if and only if $s \in [0,1]$, operator convex if and only if $s \in [-1,0] \cup [1,2]$. Other examples are

- $\log t$ is operator monotone on $(0,\infty)$.
- $t\log t$ is operator convex on $[0,\infty)$ with $0\log 0 = 0$.
- e^t is neither 2-monotone nor 2-convex on $(-\infty,\infty)$.

Kubo and Ando developed an axiomatic theory concerning operator connections and operator means for pairs of positive operators ([8]).

Definition 2.1. A binary operation σ defined by;

$$\sigma : (A,B) \in B(H)^+ \times B(H)^+ \mapsto A\sigma B \in B(H)^+$$

is called an *operator connection*, if the following properties are fulfilled.

(i) $A \leq B$ and $C \leq D$ imply $A\sigma C \leq B\sigma D$;
(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$ for all $C \in B(H)^+$;
(iii) $A_n \searrow A$ and $B_n \searrow B$ imply $(A_n\sigma B_n) \searrow (A\sigma B)$.

A *operator mean* is an operator connection with normalization condition.

(iv) $1\sigma 1 = 1$.

They showed that there exists an affine order isomorphism from the class of operator connections onto the class of positive operator monotone functions by

$$\begin{aligned} &\sigma \mapsto f_\sigma(t) = 1\sigma(t1) \quad (t > 0), \\ &f \mapsto A\sigma_f B = A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2} \end{aligned}$$

for $A, B \in B(H)^{++}$.

Remark 2.2. It is well-known that if $f : (0, \infty) \to (0, \infty)$ is operator monotone, then the transpose $f^{'}(t) = tf(\frac{1}{t})$, the adjoint $f^*(t) = \frac{1}{f(\frac{1}{t})}$, the dual $f^{\perp}(t) = \frac{t}{f(t)}$ are also operator monotone. Indeed, let σ be an operator connection corresponding to operator monotone f. Note that $x \to x^{-1}$ is operator convex and $x \to -x^{-1}$ operator monotone, then the dual $f^{\perp}(t) = \frac{t}{f(t)}$ and the adjoint $f^*(t) = \frac{1}{f(\frac{1}{t})}$ are also operator monotone. Moreover, $f^{'}(t) = tf(\frac{1}{t})$ is the corresponding function of the operator connection σ' ($A\sigma' B = B\sigma A$) ([8, Lemma 4.1]). Thus f' is also operator monotone.

Furthermore, it is also known that f is operator monotone decreasing if and only if it is operator convex and numerically non-increasing.

3 λ-weighted and operator convexity

Since $1 \leq 1\sigma t \leq t$ for all $t \geq 1$, we have

$$\left.\frac{d(1\sigma t)}{dt}\right|_{t=1} \leq \lim_{t \to 1^+} \frac{t-1}{t-1} = 1.$$

Thus, we have $\left.\frac{d(1\sigma t)}{dt}\right|_{t=1} \in [0, 1]$ (Cf. [2]).

Definition 3.1. Let $\lambda \in [0, 1]$. An operator mean σ is called λ-*weighted* if

$$\left.\frac{d(1\sigma t)}{dt}\right|_{t=1} = \lambda$$

and σ is called non-trivial if the weighted of σ is in $(0, 1)$.

Remark 3.2. If σ is λ-weighted, then $!_\lambda \leq \sigma \leq \nabla_\lambda$ ([8]). It is enough to consider the case which σ is symmetric (*i.e.* $\lambda = \frac{1}{2}$). This goes as following relation,

$$\frac{2x}{1+x} \leq \frac{1+t}{2}\left\{\frac{x}{x+t} + \frac{x}{xt+1}\right\} \leq \frac{1+x}{2}$$

for $x, t > 0$.

In the rest of the paper, we consider a continuous function ψ satisfying

$$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B) \tag{3.1}$$

for all $A, B \in B(H)^{++}$ and for a certain operator mean σ.

The following proposition is the characterization of function ψ which satisfies (3.1).

Proposition 3.3. *Let ψ be a non-negative continuous function on $(0, \infty)$. Then the following are equivalent:*

(1) ψ is operator convex;

(2) $\psi(A\nabla_\lambda B) \leq \psi(A)\nabla_\lambda\psi(B)$ for all $A, B \in B(H)^{++}$ and for all $\lambda \in (0, 1)$;

(3) $\psi(A\nabla_\lambda B) \leq \psi(A)\nabla_\lambda\psi(B)$ for all $A, B \in B(H)^{++}$ and for some $\lambda \in (0, 1)$;

(4) $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ for all $A, B \in B(H)^{++}$ and for some $\lambda \in (0, 1)$ and for some non-trivial operator mean σ.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) are trivial from the definition of operator convex. (4) $\Rightarrow$ (1) goes as follows. Consider the sequences $A_0 := A$, $B_0 := B$, $A_n := (A_{n-1}\nabla_{1-\lambda}B_{n-1})\nabla_\lambda(A_{n-1}\nabla_\lambda B_{n-1})$, $B_n := A + B - A_n$ for $n \geq 1$. By the assumption and simple calculation, ψ is operator convex.

From the above result, it is natural to assume that ψ which satisfies (3.1) is operator convex.

Proposition 3.4. *For $\lambda \in (0, 1)$, let ψ be a non-negative, non-constant, continuous function on $(0, \infty)$ and let σ be a non-trivial operator mean. Suppose that*

$$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$$

for all $A, B \in B(H)^{++}$. Then, σ is λ-weighted.

To show above argument, we consider the following lemma.

Lemma 3.5. *For $\lambda \in [0,1]$, let ψ be a non-negative continuous function on $(0,\infty)$ with a non-zero derivative at 1 and let σ be a non-trivial operator mean. Suppose that*

$$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$$

for all $A, B \in B(H)^{++}$. Then, σ is λ-weighted.

From this lemma, we only consider the case that ψ has a zero derivative at 1 to show Proposition 3.4. We may assume that $\psi(1) = 1$ by scalar multiple. Put $\phi(t) := \psi(t+1) - 1$ and $\gamma := \left.\frac{d(1\sigma t)}{t}\right|_{t=1}$. By showing that ϕ and ∇_γ satisfy the assumption of Lemma 3.5, Proposition 3.4 is proved.

In conclusion, the following corollary is obtained.

Corollary 3.6. *For $\lambda \in (0,1)$, let ψ be a non-constant, non-negative, continuous function on $(0,\infty)$ and let $\Gamma_\lambda(\psi)$ be the set of all non-trivial operator means σ such that inequality (3.1) holds for all $A, B \in B(H)^{++}$. Then, ψ is an operator convex function if and only if*

$$\{\sigma \mid !_\lambda \leq \sigma \leq \nabla_\lambda\} \supseteq \Gamma_\lambda(\psi) \supseteq \{\nabla_\lambda\}.$$

In the following corollary, a characterization of operator concave is also given by using that $\frac{1}{\phi(t)}$ is non-constant operator convex with a non-zero derivative at 1.

Corollary 3.7. *For $\lambda \in (0,1)$, let ϕ be a positive operator concave function on $(0,\infty)$ with non-zero derivative at 1 and $\phi(1) = 1$ and let σ be a non-trivial operator mean. Then, the following are equivalent:*

(1) σ is λ-weighted;

(2) $\phi(A)\sigma\phi(B) \leq \phi(A\nabla_\lambda B)$ for all $A, B \in B(H)^{++}$;

(3) $\phi^*(A!_\lambda B) \leq \phi^*(A)\sigma^*\phi^*(B)$ for all $A, B \in B(H)^{++}$, where $\phi^*(x) = (\phi(x^{-1})^{-1}$.

4 Characterization of operator convex functions

The following is a weighted version of ([1, Theorem 2.1]).

Proposition 4.1. *For $\lambda \in (0,1)$, let ψ be a non-negative continuous function on $(0,\infty)$. Then, the following conditions are equivalent:*

(1) ψ *is operator monotone decreasing;*

(2) $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ *for all* $A, B \in B(H)^{++}$ *and for all* λ*-weighted operator means* σ*;*

(3) $\psi(A\nabla_\lambda B) \leq \psi(A)\#_\lambda\psi(B)$ *for all* $A, B \in B(H)^{++}$*;*

(4) $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ *for all* $A, B \in B(H)^{++}$ *and for some* λ*-weighted operator mean* $\sigma \neq \nabla_\lambda$*,*

where $A\#_\lambda B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$.

Combining the above results, our main theorem is obtained:

Theorem 4.2 (Main result)**.** *For* $\lambda \in (0, 1)$*, let* ψ *be a non-constant, non-negative, continuous function on* $(0, \infty)$ *and let* $\Gamma_\lambda(\psi)$ *be the set of all non-trivial operator means* σ *such that the inequality*

$$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$$

holds for all $A, B \in B(H)^{++}$.
Then, the following holds:

(1). ψ *is a decreasing operator convex function if and only if*

$$\Gamma_\lambda(\psi) = \{\sigma \mid !_\lambda \leq \sigma \leq \nabla_\lambda\}.$$

(2). ψ *is an operator convex function which is not a decreasing function if and only if*

$$\Gamma_\lambda(\psi) = \{\nabla_\lambda\}.$$

By this characterization, when an operator mean $\sigma \in \Gamma_\lambda(\psi)$ is given we can determine whether ψ is decreasing or non-decreasing operator convex.

It is known that a non-negative operator convex function ψ on $[0, \infty)$ with $\psi(0) = 0$ and $\psi(1) = 1$ is strictly increasing. Therefore, the following is a direct result of the preceding theorem.

Corollary 4.3. *Let* $\lambda \in (0, 1)$*, and let* σ *be a non-trivial operator mean. Suppose that* ψ *is a non-negative operator convex function on* $[0, \infty)$*, with* $\psi(0) = 0$ *and* $\psi(1) = 1$*. Then, the following are equivalent:*

1. $\sigma = \nabla_\lambda$*;*

2. $\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ *for all* $A, B \in B(H)^{++}$.

Remark 4.4. In Theorem 4.2, the first statement implies the second one and can be proven using Corollary 4.3 and the arguments as in the proof of [1, Theorem 2.1]. Thus, these three statements (two statements in Theorem 4.2 and Corollary 4.3) are equivalent.

5 2-convex functions

If ψ is a non-negative 2-convex function on $[0,\infty)$ with $\psi(0) = 0$, then ψ is a C^2-function on $(0,\infty)$, by ([7]) (Cf. [5, Theorem 2.4.2]). Recall that ψ is said to be 2-convex if for all $A, B \in M_2(\mathbf{C})^{++}$ and $\lambda \in [0,1]$ $\psi(\lambda A + (1-\lambda)B) \leq \lambda\psi(A) + (1-\lambda)\psi(B)$. Moreover, ψ is non-constant, strictly monotone increasing on $(0,\infty)$. Indeed, by ([11, Theorem 2.2]) there exists a monotone function f on $(0,\infty)$, such that $\psi(t) = tf(t)$. Then, for any $0 < x_1 < x_2$, we have

$$\begin{aligned}\psi(x_1) &= x_1 f(x_1) \leq x_1 f(x_2)\\ &< x_2 f(x_2) = \psi(x_2).\end{aligned}$$

Using this, we present an extension of Corollary 4.3.

Proposition 5.1. *Let $\lambda \in (0,1)$, and let σ be a non-trivial operator mean. Suppose that ψ is a non-negative operator 2-convex function on $[0,\infty)$, with $\psi(0) = 0$ and $\psi(1) = 1$. Then, the following are equivalent:*

1. *$\sigma = \nabla_\lambda$;*
2. *$\psi(A\nabla_\lambda B) \leq \psi(A)\sigma\psi(B)$ for all positive definite 2×2 matrices A, B.*

Similarly, we have the following characterization of the λ-weighted harmonic mean.

Proposition 5.2. *Let ψ be a non-negative continuous function on $[0,\infty)$ with $\psi(1) = 1$ and $\lim_{x\to\infty}\psi(x) = +\infty$, and assume that $\lambda \in (0,1)$. If a non-trivial operator mean σ satisfies*

$$\psi(A!_\lambda B) \geq \psi(A)\sigma\psi(B)$$

for all positive definite 2×2 matrices A, B, then $\sigma = !_\lambda$.

6 Questions

In this article, we gave a characterization of operator means which satisfy (3.1) for a non-constant, non-negative, continuous function ψ on $(0,\infty)$. We also need to consider a characterization for more general case (Question 1). We also gave a characterization of operator convex when operator means which satisfy (3.1) were given. For this characterization as well, more general characterization should be given (Question 2). Furthermore, we need to consider the evaluation of relative entropy using operator mean described in introduction.

Questions

1. Fix a non-negative continuous function ψ with some conditions, suppose that
$$\psi(A)\sigma\psi(B) \geq \psi(A\sigma B)$$
then $\sigma = ?$

2. Fix an operator mean σ, suppose that
$$\psi(A)\sigma\psi(B) \geq \psi(A\sigma B)$$
then $\psi = ?$

References

[1] T. Ando, F. Hiai, *Operator log-convex functions and operator means.* Math. Ann. 350 (2011), no. 3, 611630.

[2] J. I. Fujii, *Operator means and Range inclusion*, Linear Algebra Appl. 170 (1992), 137–146.

[3] S. Furuichi, N. Minculete, F. Mitroi, *Some inequalities on generalized entropies.* J. Inequal. Appl. 2012, 2012:226, 16 pp.

[4] F. Hansen and G. K. Pedersen, *Jensen's inequality for operator and Löwner's theorem*, Math. Ann. 258 (1982) 229-241.

[5] F. Hiai, *Matrix analysis: matrix monotone functions, matrix means, and Majorization*, Interdiscip. Inform. Sci. vol 16 (2010), no. 2, 139–248.

[6] F. Hiai and D. Petz, *Introduction to matrix analysis and applications*, Universitext, Springer, New Delhi, 2014.

[7] F. Kraus, *Über Konvexe Mathtrixfunctiouen*, Math. Z. 41(1936) 18–42.

[8] F. Kubo, T. Ando, *Means of positive linear operators.* Math. Ann. 246 (1979/80), no. 3, 205224.

[9] K. Löwner, *Über monotone matrixfunktionen*, Math. Z. **38** (1934) 177–216.

[10] C. P. Niculescu and L. -E. Persson, *Convex functions and their applications. A contemporary approach*, CMS Books in Mathematics vol. 23. Springer, New York, 2006.

[11] H. Osaka, J. Tomiyama, *Double piling structure of matrix monotone functions and of matrix convex functions*. Linear Algebra Appl. 431 (2009), no. 10, 18251832.

[12] H. Osaka, Y. Tsurumi, S. Wada, *Characterization of operator convex functions by certain operator inequalities*, to appear in Mathematical Inequalities & Applications.

Graduate school of science and engineering
Ritsumeikan University
Shiga 525-8577
JAPAN
ra0006hk@ed.ritsumei.ac.jp

立命館大学大学院　理工学研究科　鶴見幸大